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Speeding strings

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Abstract: There is a class of single trace operators in $\mathcal{N}=4$ Yang-Mills theory which are related by the AdS/CFT correspondence to classical string solutions. Interesting examples of such solutions corresponding to periodic trajectories of the Neumann system were studied recently. In our paper we study a generalization of these solutions. We consider strings moving with large velocities. We show that the worldsheet of the fast moving string can be considered as a perturbation of the degenerate worldsheet, with the small parameter being the relativistic factor $\sqrt{1-v^2}$. The series expansion in this relativistic factor should correspond to the perturbative expansion in the dual Yang-Mills theory. The operators minimizing the anomalous dimension in the sector with given charges correspond to periodic trajectories in the mechanical system which is closely related to the product of two Neumann systems.

Keywords: AdS-CFT and dS-CFT Correspondence, Differential and Algebraic Geometry, Integrable Equations in Physics.

Contents

1. Introduction

The operators with the large spin or large R charge in $\mathcal{N} = 4$ Yang-Mills theory are of a special interest for the AdS/CFT correspondence. The study of these operators initiated in [1]–[4] gives the most convincing evidence for the validity of the Maldacena conjecture. The R charge of the operator is specified by the representation of the R-symmetry algebra, or equivalently by the highest weight $[J_1, J_2, J_3]$. It was shown in [3] that the operators with $J_1 \gg J_2, J_3$ (now known as BMN operators) have anomalous dimensions which can be represented as power series in λ/J_1^2 where λ is the t'Hooft coupling constant. The authors of [3] derived this series expansion from the string theory, and later it was reproduced in the Yang-Mills perturbation theory. Recently more general operators with J_1, J_2 and J_3 of the same order of magnitude and also with large spins were studied in the series of papers [5]–[16]. Unlike the BMN operators, these operators cannot be considered as small deformations of the BPS operators. But their anomalous dimension is of the order λ/J, and there is a conjecture that in the limit of large charges one can compute it from the energy of the corresponding string theory state in the large radius $AdS_5 \times S^5$.

It turns out that the corresponding string theory state is a single semiclassical string. The simplest case to consider is a spin zero operator. The worldsheet of the corresponding string is a product of a timelike geodesic in AdS_5 and a rotating contour in S^5. The authors of [11] suggested the following ansatz for this solution:

$$x_1 + ix_2 = x_1(\sigma)e^{i\frac{w_1}{k}t}, \qquad x_3 + ix_4 = x_2(\sigma)e^{i\frac{w_2}{k}t}, \qquad x_5 + ix_6 = x_3(\sigma)e^{i\frac{w_3}{k}t}. \quad (1.1)$$

Here $x_1, \ldots, x_6$ are the coordinates on a sphere S^5 subject to the constraint $x_1^2 + \ldots + x_6^2 = 1$, and t is a length parameter on the timelike geodesic in AdS_5. Substitution of this ansatz into the string worldsheet action leads to the one-dimensional mechanical system with the lagrangian:

$$L = \frac{1}{2} \sum_{i=1}^{3} ((\partial_\sigma x_i)^2 - w_i^2 x_i^2) \tag{1.2}$$

where x_i are restricted to a sphere: $\sum_{i=1}^3 x_i^2 = 1$. The corresponding R-charges are

$$J_i = \sqrt{\lambda} w_i \int \frac{d\sigma}{2\pi} x_i^2(\sigma) \,. \tag{1.3}$$

For comparison to the field theory computation the most interesting case is when the momenta J_i are very large. It was noticed in [17] that this limit corresponds to the string moving very fast, and the induced metric on the string worldsheet becoming nearly degenerate. In some sense the Yang-Mills perturbation series for such operators should correspond to the expansion in the powers of the relativistic factor $\sqrt{1 - v^2/c^2}$.

Motivated by this observation we study in this paper classical strings in $AdS_5 \times S^5$ moving with large velocities. Such solutions correspond to the Yang-Mills operators with large charges and spins. We will be interested in the operators which have very large charges J_a and finite but small ratios λ/J_a^2. We will pick a special combination of charges which we will denote Q_{tot} and define

$$\epsilon^2 = \frac{\lambda}{Q_{\text{tot}}^2} \,. \tag{1.4}$$

Suppose that we have a uniform definition of the operator for all values of the coupling constant λ, and the operator corresponds to the classical string solution in $AdS_5 \times S^5$. When we change the coupling constant the shape of the worldsheet changes. Of course, the radius of $AdS_5 \times S^5$ also changes as $\lambda^{1/4}$, but this is just the overall coefficient in front of the metric. Given a classical string solution in $AdS_5 \times S^5$ we actually get the whole one-parameter family of classical solutions by varying the coupling constant λ. We will call this family of string worldsheets λ-*family*. In the regime we are interested in it is convenient to parametrize this λ-family by ϵ. Let $\Sigma(\epsilon)$ be the worldsheet of the string corresponding to our operator at the coupling constant $\lambda = Q_{\text{tot}}^2 \epsilon^2$. We will be interested in the class of operators such that the corresponding worldsheet $\Sigma(\epsilon)$ has a well-defined limit when $\epsilon \to 0$. In this limit it becomes a null-surface $\Sigma(0)$, a surface with the degenerate metric ruled by the light rays. Moreover, this null-surface naturally comes with an additional structure. This additional structure is a function $\sigma : \Sigma(0) \to S^1$ constant along the light rays. In other words, if we think of $\Sigma(0)$ as a collection of the light rays, then this collection is a one-parameter family, parametrized by σ. This function σ is determined by the shape of $\Sigma(\epsilon)$ for very small but nonzero ϵ. It is defined modulo the overall shift (if $\sigma_1 = \sigma_2 +$const, then σ_1 and σ_2 should be considered equivalent.) The definition requires the choice of a particular combination of symmetries (corresponding to the charge Q_{tot}.) In fact $\frac{\sqrt{\lambda}}{\epsilon} d\sigma$ is the density of the charge Q_{tot} on the worldsheet $\Sigma(\epsilon)$ in the limit $\epsilon \to 0$.

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We conjecture that the λ-family (and the corresponding YM operator) is in fact uniquely determined by the null-surface $\Sigma(0)$ and the real function σ. Indeed, to specify the λ-family it should be enough to specify the worldsheet at some finite value of λ. The worldsheet is parametrized by 16 real functions of one real variable, 8 functions specifying the shape of the string at time zero and 8 functions specifying the velocities. But $\Sigma(0)$ is specified by 15 real functions, plus one real function σ, therefore we get also 16 real functions. This counting of the parameters leads us to the conjecture that there is a correspondence between the non-degenerate extremal surfaces and the degenerate surfaces with a function σ constant on the light rays. We must stress that this construction uses the assumption that there is a uniform definition of the Yang-Mills operator for all values of the coupling constant λ.

Let us summarize. The Yang-Mills operators (of certain type) are in one-to-one correspondence with the pairs $(\Sigma(0), \sigma)$ where $\Sigma(0)$ is a null-surface in $AdS_5 \times S^5$ and σ is a function from $\Sigma(0)$ to S^1 constant on the light rays. On the other hand, the operator determines a family of contours $\Sigma(\epsilon)$ where $\epsilon^2 = \lambda/J^2$, such that the limit when $\epsilon \to 0$ is $\Sigma(0)$. This means that for each null-surface $\Sigma(0)$ with a function σ there should be a preferred family of deformations $\Sigma(\epsilon)$.

In this paper we will focus on a special class of operators for which a particular combination of their charges and their conformal dimension is small (compared to other charges.) In perturbation theory such operators are defined by the requirement that their engineering dimension is equal to certain linear combination of their spins and R-charges. The combination of the dimension and charges which is small corresponds in the AdS picture to certain lightlike Killing vector V of $AdS_5 \times S^5$. In the limit of infinite velocities the worldsheet becomes a null-surface spanned by the integral curves of V (which are automatically null-geodesics.) A null surface spanned by the integral curves of V is specified by a curve in the coset space $(AdS_5 \times S^5)/V$ such that the tangent direction to the curve is orthogonal to V. Moreover, we have a function σ constant on the light rays which we can use to parametrize the curve. Therefore we get a map $C : S^1 \to (AdS_5 \times S^5)/V$. The charge corresponding to V is given in the first order in ϵ by the "action functional"

$$\epsilon\sqrt{\lambda} \int_{S^1} d\sigma\, g_{ij}(C) \frac{dC^i(\sigma)}{d\sigma} \frac{dC^j(\sigma)}{d\sigma} \tag{1.5}$$

where g_{ij} is the metric on $(AdS_5 \times S^5)/V$. This charge corresponds to the one loop anomalous dimension of the operator. The Killing vector fields other than V correspond to the charges which go to infinity in the limit $\epsilon \to 0$. These "large" charges are of the form $\frac{\sqrt{\lambda}}{\epsilon} \int d\sigma F(C)$ with certain function F. The particular combination of charges which we denoted $Q_{\rm tot}$ corresponds to $F(C) \equiv 1$.

Therefore the operator defines formally a closed trajectory of the particle moving in $(AdS_5 \times S^5)/V$ under the constraint that the velocity is orthogonal to V. The action on this trajectory corresponds to the one-loop anomalous dimension of the operator. The trajectory does not have to satisfy any equations of motion.

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Perhaps one could define a measure on the space of contours in $(AdS_5 \times S^5)/V$, which would compute how many operators have the anomalous dimension in a given interval. One concrete question is what is the minimal one-loop anomalous dimension in the sector with the given charges. At the level of one loop this problem reduces to finding periodic trajectories of certain C. Neumann type mechanical system which is a natural generalization of the one considered in [11]. We explain the details in section 3. The Neumann systems are closely related to integrable spin chains — see the discussion in [18].

It would be interesting to understand in general when two string worldsheets belong to the same λ-family. For the operators which we are discussing in this paper the dependence of the string worldsheet on λ is of the crucial importance. Indeed, the string worldsheet action is proportional to $R^2/\alpha' \sim \sqrt{\lambda}$, and because of that one could naively guess that the anomalous dimension of the operator is of the order $\sqrt{\lambda}$ in the strong coupling regime. But this is wrong in the case we are considering precisely because the shape of the string worldsheet itself depends on λ, and in such a way that the V-charge of the λ-dependent worldsheet is proportional to λ rather than $\sqrt{\lambda}$.

If we want to know the anomalous dimension of the operator as a function of λ, it is not enough to know just the string worldsheet corresponding to this operator. We have to know the whole λ-family. But there are some questions which can be answered without the knowledge of the λ-families. We can ask what is the minimal anomalous dimension of the operator in the sector with the given charges, for the given value of the coupling constant. In perturbation theory, the answer will be a series in λ. If we want to answer this question in the strong coupling regime using the AdS/CFT correspondence we have to find the string worldsheet which has a minimal energy for given charges. For different values of the coupling (different radii of the AdS space) we will get different worldsheets. Therefore, we will get some family of worldsheets — but it is not guaranteed that this will be a λ-family! Indeed, there is no apriori reason why the same operator would minimize the anomalous dimension in the sector with the given charges, for different values of the coupling constant.

The plan of the paper. In section 2 we will give a definition of the null-surface and explain that the null surface is the limit of the family of ultrarelativistic extremal surfaces. We will also explain how to compute the conserved charges in this limit. In section 3 we will study the ultrarelativistic surfaces in $AdS_5 \times S^5$. We will show that the operators minimizing the one loop anomalous dimension for given charges correspond to the periodic trajectories of the gauged Neumann system. We briefly review the basic properties of the Neumann systems. We explain that the space of periodic trajectories consists of several branches corresponding to the possible degenerations of the invariant tori.

After this paper was completed we have received the preprint [19] which has an overlap with our paper.

Note added in the revised version We have made a mistake in the original version of this paper which lead us to the conclusion that the correspondence between the operators and the pairs $(\Sigma(0), \sigma)$ can not be one-to-one. We claimed that in order to specify the λ-family, one has to know besides $(\Sigma(0), \sigma)$ also $\Sigma(\epsilon)$ to the first order in ϵ. We realized that we made a mistake studying the recent preprint [24] which contains the construction of $\Sigma(\epsilon)$ to the first order in ϵ from the known $\Sigma(0)$ and σ, for $\Sigma(0)$ generated by the orbits of V.

We have also learned about the papers [23] where the degenerate surfaces were discussed in the context of Frolov-Tseytlin solutions. The authors of [23] studied ultrarelativistic strings in the backgrounds with the B-field, as well as ultrarelativistic membranes.

2. Null surfaces and their deformations

2.1 Definition of a null surface

Consider a two-dimensional surface Ξ embedded in the metric space M with Minkowski signature $(1, d-1)$, $d > 2$. We will say that Ξ is *isotropic* if the induced metric is degenerate. This means that for every point $x \in \Xi$ the tangent space $T_x\Xi$ has a null-vector $v(x)$ which is orthogonal to all other vectors in $T_x\Xi$. The vectors in $T_x\Xi$ which are not parallel to $v(x)$ are all space-like. The integral curves of the vector field $v(x)$ will be called null-curves.

We will call an isotropic surface Ξ a *null-surface* if its null-curves are light rays (null geodesics) in M.

For example, let us consider a future light cone of a point $x_0 \in M$. It is generated by the light rays emitted at x_0. If we choose any one-parameter family of the light rays emitted at x_0, then this family will sweep a null surface. More general examples can be obtained in the following way. Consider a spacelike curve C. For each point $x \in C$ pick a light ray going through x, such that the direction of this light ray at the point x is orthogonal to T_xC. The resulting one-parameter family of light rays will be a null surface.

We are interested in null-surfaces because they can be thought of as worldsheets of ultrarelativistic strings. The classical equations of motion imply that the string worldsheet is an extremal surface. If the string moves very fast the surface becomes isotropic. Moreover, one can see that the isotropic limit of the extremal surface should be a null surface. Indeed, let us introduce on the string worldsheet the coordinates ξ^+, ξ^-, so that the induced metric is $\rho(\xi^+, \xi^-)d\xi^+ d\xi^-$. The condition that the surface is extremal is

$$\frac{D}{\partial\xi^+}\frac{\partial x^\mu}{\partial\xi^-} = 0 \tag{2.1}$$

where $x^\mu(\xi^+, \xi^-)$ are the embedding functions. In the limit when the surface becomes isotropic, the two null-directions $\partial/\partial\xi^+$ and $\partial/\partial\xi^-$ coincide, and eq. (2.1) implies that the limiting null-curves are geodesics. We will argue that any null-surface can be obtained as a limit of a family of extremal surfaces.

2.2 Ultrarelativistic surfaces

We are interested in the case when the space-time admits a light-like Killing vector field V. We will require that the light rays forming the null-surface $\Sigma(0)$ are the integral curves of V. Consider a family of surfaces $\Sigma(\epsilon)$ which approach $\Sigma(0)$ when $\epsilon \to 0$, in the sense that the coordinate difference between $\Sigma(0)$ and $\Sigma(\epsilon)$ is of the order ϵ^2. We will now describe a special choice of coordinates on $\Sigma(\epsilon)$.

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Suppose that $\Sigma(\epsilon)$ is topologically a cylinder. Pick a closed space-like contour $C_0 \subset \Sigma(\epsilon)$ For each point $x \in C_0$ take a direction in $T_x\Sigma(\epsilon)$ which is orthogonal to $T_xC_0 \subset T_x\Sigma(\epsilon)$. Since $\Sigma(\epsilon)$ is close to $\Sigma(0)$ this direction is close to the direction of V. Therefore we can choose a vector $u(x) \in T_x\Sigma(\epsilon)$ which is orthogonal to T_xC_0 and $u(x) = V(x) + O(\epsilon^2)$. We have $(u(x), u(x)) \simeq \epsilon^2$. Let us introduce the parametrization σ of C_0 in the following way:

$$\left(\frac{\partial x(\sigma)}{\partial\sigma}, \frac{\partial x(\sigma)}{\partial\sigma}\right) = -\frac{1}{\epsilon^2}(u(x), u(x))\,. \tag{2.2}$$

The coordinates τ and σ on $\Sigma(0)$ are determined by the conditions on the embedding functions $x(\tau,\sigma)$:

$$(\partial_\tau x, \partial_\tau x) = -\epsilon^2(\partial_\sigma x, \partial_\sigma x)\,, \qquad (\partial_\tau x, \partial_\sigma x) = 0 \tag{2.3}$$

$$x(0,\sigma) = x(\sigma) \in C_0\,. \tag{2.4}$$

The first equation uniquely determines $\partial_\tau x(\tau_0,\sigma) \in T_{x(\tau_0,\sigma)}\Sigma(\epsilon)$ from the contour $x(\tau_0,\sigma)$ for a fixed $\tau = \tau_0$. Therefore it is an "evolution equation" for the contour on the worldsheet. The second equation determines the initial condition — the contour at $\tau = 0$. The equation for the extremal surface in these coordinates is $D_\tau\partial_\tau x = \epsilon^2 D_\sigma\partial_\sigma x$. It implies that at the zeroth order in ϵ, $x(\tau,\sigma_0)$ for a fixed σ_0 is a null geodesic. Therefore $\partial_\tau x = V + O(\epsilon^2)$ not only on the initial curve C_0 but everywhere on the worldsheet.

To summarize, we have chosen the coordinates σ,τ on $\Sigma(\epsilon)$ such that the embedding functions $x_\epsilon(\sigma,\tau)$ satisfy the constraints

$$\left(\frac{\partial x_\epsilon}{\partial\tau}, \frac{\partial x_\epsilon}{\partial\tau}\right) + \epsilon^2\left(\frac{\partial x_\epsilon}{\partial\sigma}, \frac{\partial x_\epsilon}{\partial\sigma}\right) = 0$$

$$\left(\frac{\partial x_\epsilon}{\partial\tau}, \frac{\partial x_\epsilon}{\partial\sigma}\right) = 0 \tag{2.5}$$

and the equations of motion

$$\frac{1}{\epsilon}D_\tau\partial_\tau x_\epsilon - \epsilon D_\sigma\partial_\sigma x_\epsilon = 0\,. \tag{2.6}$$

There is a residual gauge invariance; the constraints (2.5) and the equations (2.6) are preserved by the infinitesimal vector fields

$$[f_L(\sigma+2\epsilon\tau) + f_R(\sigma-2\epsilon\tau)]\frac{\partial}{\partial\tau} + \epsilon\,[f_L(\sigma+2\epsilon\tau) - f_R(\sigma-2\epsilon\tau)]\frac{\partial}{\partial\sigma}\,. \tag{2.7}$$

We want to study the solutions to the equations (2.6) and the constraints (2.5). Let us consider the following ansatz:

$$x_\epsilon(\sigma,\tau) = x_0(\sigma,\tau) + \epsilon^2\eta_1(\sigma,\tau) + \epsilon^4\eta_2(\sigma,\tau) + \cdots\,. \tag{2.8}$$

This ansatz is preserved by the vector fields (2.7) with $f_L = f_R$. The equations of motion and constraints for x imply:

$$\frac{D^2\eta_1}{\partial\tau^2} + R\left(\frac{\partial x_0}{\partial\tau}, \eta_1\right)\frac{\partial x_0}{\partial\tau} = \frac{D}{\partial\sigma}\frac{\partial x_0}{\partial\sigma} \tag{2.9}$$

$$(D_\sigma\eta_1, \partial_\tau x_0) + (D_\tau\eta_1, \partial_\sigma x_0) = 0 \tag{2.10}$$

$$(D_\tau\eta_1, \partial_\tau x_0) = -\frac{1}{2}(\partial_\sigma x_0\,, \partial_\sigma x_0) \tag{2.11}$$

where $R(\xi,\eta) = -\nabla_\xi\nabla_\eta + \nabla_\eta\nabla_\xi + \nabla_{[\xi,\eta]}$ is the curvature tensor. The first equation detremines $\eta_1(\sigma,\tau)$ from the initial conditions $\eta_1(\sigma,0)$ and $\partial_\tau|_{\tau=0}\eta_1(\sigma,\tau)$ which should satisfy the constraints (2.10) and (2.11). The higher coefficients η_i satisfy the similar equation and constraints. But as we have explained in the Introduction, we expect that η_i are actually determined from x_0. This is because we want the solutions for different ϵ to correspond to the same operator in the gauge theory but at the different values of the coupling constant. For the same reason we have omitted in our ansatz (2.8) the odd powers of ϵ. The parameter ϵ is proportional to $\sqrt{\lambda}$; we hope that the string solution has an expansion in integer powers of λ.

2.3 Conserved quantities

The string worldsheet action is

$$S = \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau \left[\frac{1}{\epsilon}(\partial_\tau x, \partial_\tau x) - \epsilon(\partial_\sigma x, \partial_\sigma x)\right]. \tag{2.12}$$

Suppose that the spacetime admits the Killing vector field $W^\mu(x)$. The corresponding conserved charge is given by the integral over the spacial slice of the string worldsheet of the closed form

$$*j = \frac{\sqrt{\lambda}}{4\pi}\left(\frac{1}{\epsilon}W_\mu(x)\partial_\tau x^\mu d\sigma + \epsilon W_\mu(x)\partial_\sigma x^\mu d\tau\right). \tag{2.13}$$

On the ansatz (2.8):

$$\begin{aligned} *j = \frac{\sqrt{\lambda}}{4\pi}\Big[&\frac{1}{\epsilon}W_\mu(x_0)\partial_\tau x_0^\mu\, d\sigma + \epsilon\Big(\eta_1^\nu\nabla_\nu W_\mu(x_0)\,\partial_\tau x_0^\mu\, d\sigma + W_\mu(x_0)D_\tau\eta_1^\mu\, d\sigma + \\ &+ W_\mu(x_0)\partial_\sigma x_0^\mu\, d\tau\Big) + O(\epsilon^3)\Big]. \end{aligned} \tag{2.14}$$

Let us consider two conserved charges, the charge corresponding to V and the charge corresponding to some other Killing vector field $U \neq V$. The charge corresponding to U is

$$Q_U = \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{4\pi}\int d\sigma U_\mu\partial_\tau x_0^\mu + O(\epsilon) = \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{4\pi}\int d\sigma\, U_\mu V^\mu + O(\epsilon) \tag{2.15}$$

and the charge corresponding to V is

$$Q_V = \epsilon\frac{\sqrt{\lambda}}{4\pi}\int d\sigma\, \partial_\tau x_0^\mu D_\tau\eta_{1\mu} = -\epsilon\frac{\sqrt{\lambda}}{8\pi}\int d\sigma\, \partial_\sigma x_\mu\partial_\sigma x^\mu + O(\epsilon^3) \tag{2.16}$$

where we have taken into account the constraint (2.11). From eq. (2.15)

$$\epsilon = \frac{\sqrt{\lambda}}{4\pi Q_U}\left[\int d\sigma\, U_\mu V^\mu + O(\epsilon^2)\right] \tag{2.17}$$

$$= \frac{\sqrt{\lambda}}{4\pi Q_U}\left[\int d\sigma\, U_\mu V^\mu + O\left(\frac{\lambda}{Q_U^2}\right)\right]. \tag{2.18}$$

Then (2.16) implies that in the limit $Q_U \to \infty$

$$Q_V = \frac{1}{32\pi^2}\frac{\lambda}{Q_U}\left[\int d\sigma\, U_\mu V^\mu \int d\sigma\, (\partial_\sigma x_0, \partial_\sigma x_0) + \cdots\right] \tag{2.19}$$

where dots stand for the subleading terms. If both Q_U and λ go to infinity in such a way that λ/Q_U^2 is finite but small, then these subleading terms are the power series in λ/Q_U^2. The coefficients of these power series explicitly depend on $\eta_k(\sigma,\tau)$.

Let us summarize what we have. At the first order in λ the V-charge (2.19) is proportional to the "action functional" for the trajectory $x_0(\sigma)$ which determines the degenerate surface $\Sigma(0)$:

$$S = \int d\sigma (\partial_\sigma x_0, \partial_\sigma x_0)\,. \tag{2.20}$$

For $\Sigma(0)$ to be degenerate, the velocity is constrained to be orthogonal to V:

$$(V, \partial_\sigma x_0) = 0\,. \tag{2.21}$$

And we have to remember that two contours $x_0(\sigma)$ which are different by the σ-dependent shift along V give the same surface $\Sigma(0)$, therefore the "gauge symmetry":

$$\delta x_0^\mu(\sigma) = \delta\phi(\sigma)\, V^\mu(x_0(\sigma)) \tag{2.22}$$

where $\delta\phi(\sigma)$ is a σ-dependent parameter.

3. Ultrarelativistic surfaces in $AdS_5 \times S^5$

3.1 Relations between the charges in the ultrarelativistic limit

The space $AdS_5 \times S^5$ is the product of a hyperboloid in $\mathbf{R}^{2+4}$ and a sphere in $\mathbf{R}^6$. We will choose a complex structure in $\mathbf{R}^{2+4}$ and $\mathbf{R}^6$ and introduce the complex coordinates Y_I , $I = 0,1,2$ in $\mathbf{R}^{2+4}$ and Z_I, $I = 1,2,3$ in $\mathbf{R}^6$. The hyperboloid and the sphere are given by the equations $|Y_0|^2 - |Y_1|^2 - |Y_2|^2 = 1$ and $|Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1$ respectively. We will denote Y_I and Z_I collectively as X_A, $A = 0,1,\ldots,5$. We will put $(X_0, X_1, X_2) = (Y_0, Y_1, Y_2)$ and $(X_3, X_4, X_5) = (Z_1, Z_2, Z_3)$. Let us also define the sign factor s_I, $(s_0, s_1, s_2) = (1,-1,-1)$.

The symmetry group of $AdS_5 \times S^5$ is SO(2,4) $\times$ SO(6). The Cartan torus is six-dimensional. We can choose it to be represented by the six Killing vector fields $U_0,\ldots,U_5$:

$$U_A.X_B = i\delta_{AB}X_B\,. \tag{3.1}$$

We will choose the light-like Killing vector field to be $V = \sum_{A=0}^5 U_A$. The corresponding charges are:

$$Q_{U_A} = \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{4\pi}\int d\sigma\, U_A^\mu V_\mu = \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{4\pi}\int d\sigma\, s_A|X_A|^2 + \cdots \tag{3.2}$$

$$\begin{aligned} Q_V &= -\epsilon\frac{\sqrt{\lambda}}{8\pi}\int d\sigma\, \partial_\sigma x_\mu \partial_\sigma x^\mu \\ &= -\epsilon\frac{\sqrt{\lambda}}{8\pi}\int_{\tau=\tau_0} d\sigma \sum_A s_A |\partial_\sigma X_A|^2 + \cdots \end{aligned} \tag{3.3}$$

where s_A is a sign: $(s_0,\ldots,s_5) = (-1,1,1,1,1,1)$. Let us introduce a special combination of charges:

$$Q_{\text{tot}} = Q_{U_0} - Q_{U_1} - Q_{U_2} = Q_{U_3} + Q_{U_4} + Q_{U_5} = \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{4\pi}\int d\sigma + \cdots\,. \tag{3.4}$$

Let us require that the period of σ is 2π:

$$\int d\sigma = 2\pi\,. \tag{3.5}$$

This condition defines ϵ in terms of $\sqrt{\lambda}/Q_{\mathrm{tot}}$. With this notation

$$Q_V = \frac{\lambda}{16\pi Q_{\mathrm{tot}}}\int d\sigma\,(\partial_\sigma x_0, \partial_\sigma x_0) + \cdots\,. \tag{3.6}$$

3.2 Gauged Neumann system

Finding the extremum of Q_V for fixed $Q_0, \ldots, Q_5$ is reduced to extremizing the functional

$$S_0[x(\sigma)] = \int d\sigma\,\partial_\sigma x_\mu \partial_\sigma x^\mu \tag{3.7}$$

for fixed $q_A = \int d\sigma |X_A|^2$ and with the constraint:

$$V_\mu(X)\partial_\sigma X^\mu = 0 \tag{3.8}$$

which follows from eq. (2.5) in the limit $\epsilon \to 0$. The solutions correspond to periodic trajectories of the mechanical system with the following lagrangian:

$$L = \sum_{A=0}^{5} s_A |\partial_\sigma X_A(\sigma)|^2 - \sum s_A \gamma_A (|X_A(\sigma)|^2 - q_A) - \Lambda(\sigma) M_{\mathrm{tot}}(\sigma) \tag{3.9}$$

where

$$M_{\mathrm{tot}}(\sigma) = i\sum_{A=0}^{5} s_A \overline{X}_A(\sigma)\overset{\leftrightarrow}{\partial}_\sigma X_A(\sigma) \tag{3.10}$$

and X_A is restricted to $AdS_5 \times S^5 \subset \mathbf{C}^{1+5}$. The first term is the action of the free particle moving on $(AdS_5 \times S^5)/V$. The other terms are fixing $q_A = \frac{1}{2\pi}\int d\sigma |X_A(\sigma)|^2$ and imposing the constraint $M_{\mathrm{tot}}(\sigma) = 0$ which follows from (3.8). The constants γ_A and the function $\Lambda(\sigma)$ are the Lagrange multipliers. The action and the constraint are invariant under the gauge transformation

$$X_A(\sigma) \mapsto e^{i\phi(\sigma)} X_A(\sigma)\,, \qquad \Lambda(\sigma) \mapsto \Lambda(\sigma) + \frac{\partial\phi(\sigma)}{\partial\sigma} \tag{3.11}$$

which corresponds to the residual diffeomorphism invariance (2.7) of the string worldsheet theory. One can choose the gauge $\Lambda(\sigma) = 0$. In this gauge the equations of motion coincide with the equations of motion of two independent conventional (not gauged) Neumann systems. Because of the constraint and the gauge symmetry this system has eight degrees of freedom, rather than ten degrees of freedom as the conventional Neumann system on $AdS_5 \times S^5$ would have.

Notice that the kinetic term $\sum_{A=0}^{5} s_A |\partial_\sigma X_A(\sigma)|^2$ is positive definite if the constraint $M_{\mathrm{tot}} = 0$ is satisfied. One can see this by choosing the gauge $\mathrm{Im}\,X_0 = 0$. And of course, the periodic trajectories we are interested in are supposed to be periodic only modulo the gauge transformation.

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3.3 Off-diagonal charges

There are 24 Killing vectors in $AdS_5 \times S^5$ which do not commute with $U_0, \ldots, U_5$. The leading expressions for these charges when $\epsilon \to 0$ are of the form $Q_{\mu\nu} = \frac{1}{\epsilon} \int d\sigma\, x_\mu(\sigma) x_\nu(\sigma)$. Suppose that the Killing vector W can be represented as $W = [V, W']$ where W' is another Killing vector. Then $(V, W) = 0$ and the charge corresponding to W is of the order ϵ. But there are symmetries which cannot be represented as a commutator of another symmetry with V. The corresponding charges are generally speaking of the order $\frac{1}{\epsilon}$. The corresponding vector fields are of the form $W_\Phi = I.\mathrm{grad}\, \Phi(X)$ where I is the complex structure and

$$\Phi(X) = \sum_{I,\bar{J}} \Phi^{I\bar{J}} Y_I \bar{Y}_{\bar{J}} + \sum_{I\bar{J}} \tilde{\Phi}^{I\bar{J}} Z_I \bar{Z}_{\bar{J}}$$

and $\Phi, \tilde{\Phi}$ are constant hermitean matrices with zeroes on a diagonal. We have $(W_\Phi, V) = \Phi$.

Suppose that the contour is a solution to the Neumann equations with $A_I \neq A_J$ for $I \neq J$ and $\tilde{A}_K \neq \tilde{A}_L$ for $K \neq L$. Then the non-diagonal charges are zero in the order $\frac{1}{\epsilon}$. Indeed, the coefficient of $\frac{1}{\epsilon}$ in the corresponding charge is computed as $\Delta_\Phi S = \int d\sigma \Phi(X(\sigma))$. Let us consider $\Delta_\Phi S$ as a small perturbation of the action of the Neumann system. In other words, take very small Φ and add $\Delta_\Phi S$ to the action. Then the value of $\Delta_\Phi S$ on a given periodic trajectory of the Neumann system can be computed as the first order correction to the action on the periodic trajectory of the perturbed system. But the first order correction to the action is actually zero. Indeed, $\Phi^{I\bar{J}}$ and $\tilde{\Phi}^{I\bar{J}}$ are hermitean matrices, therefore the potential in the perturbed system would correspond to a pair of hermitean matrices $\mathrm{diag}(A) + \Phi$, $\mathrm{diag}(\tilde{A}) + \tilde{\Phi}$. We can diagonalize them by unitary transformations. We get the Neumann system with the new coefficients $A', \tilde{A}'$. But the differences $A' - A$ and $\tilde{A}' - \tilde{A}$ are of the second order in Φ. Therefore the first order correction to the action is zero. (See appendix C of [11] for an alternative proof.)

But if the contour is arbitrary, not a solution to the nondegenerate Neumann equation, then the non-diagonal charges may be nonzero. In general we will get a pair of hermitean matrices of charges

$$q_{I\bar{J}} = \int d\sigma\, Y_I \bar{Y}_{\bar{J}}\,, \qquad \tilde{q}_{I\bar{J}} = \int d\sigma\, Y_I \bar{Y}_{\bar{J}} \tag{3.12}$$

for the string solution which does not minimize the Q_V for given diagonal charges $Q_0, \ldots, Q_5$.

3.4 Exact worldsheets

We have argued that the string solutions can be obtained as perturbations near the degenerate worldsheet with the small parameter ϵ. In [11] the solutions of the Neumann system were used to construct the exact worldsheet, rather than an expansion in a small parameter. More general Neumann system considered here can also be used to build exact string worldsheets. Consider the following ansatz:

$$Y_I(\sigma, t) = e^{iw_I t} Y_I(\sigma)\,, \qquad Z_I(\sigma, t) = e^{i\tilde{w}_I t} Z_I(\sigma) \tag{3.13}$$

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where $Y_I(\sigma)$ and $Z_I(\sigma)$ solve the Neumann system equations of motion:

$$\partial_\sigma^2 Y_I + Y_I \sum s_J |\partial_\sigma Y_J|^2 = -w_I^2 Y_I + Y_I \sum s_J w_J^2 |Y_J^2|$$

and the analogous equation for Z. The functions $Y_I(\sigma)$ and $Z_I(\sigma)$ should be periodic modulo

$$(Y_I, Z_I) \to (e^{iw_I\varphi} Y_I, e^{i\tilde{w}_I\varphi} Z_I)\,. \tag{3.14}$$

We have

$$\begin{aligned}(\partial_t x, \partial_t x) &= \sum s_I w_I^2 |Y_I|^2 - \sum \tilde{w}_I^2 |Z_I|^2 \\ (\partial_\sigma x, \partial_\sigma x) &= \sum s_I |\partial_\sigma Y_I|^2 - \sum |\partial_\sigma Z_I|^2\,.\end{aligned}$$

Let us restrict $w_I, \tilde{w}_I$ to satisfy:

$$\sum s_I w_I M_I = \sum \tilde{w}_I \tilde{M}_I\,. \tag{3.15}$$

This constraint implies that $(\partial_\sigma x, \partial_t x) = 0$. The trace of the second fundamental form in the AdS_5 direction is:

$$\frac{-w_I^2 Y_I + Y_I \sum s_J w_J^2 |Y_J^2|}{\sum s_I w_I^2 |Y_I|^2 - \sum \tilde{w}_I^2 |Z_I|^2} + \frac{\partial_\sigma^2 Y_I + Y_I \sum s_J |\partial_\sigma Y_J|^2}{\sum s_I |\partial_\sigma Y_I|^2 - \sum |\partial_\sigma Z_I|^2} \tag{3.16}$$

and the analogous expression in the S^5 direction. It is zero for the solutions which have zero energy:

$$\sum s_I |\partial_\sigma Y_I|^2 - \sum |\partial_\sigma Z_I|^2 + \sum s_I w_I^2 |Y_I|^2 - \sum \tilde{w}_I^2 |Z_I|^2 = 0 \tag{3.17}$$

which means that the ansatz solves the equations for the extremal surface. It is not true that (3.14) with σ-dependent φ is a symmetry of the action. The Neumann system with the constraint (3.15) has nine degrees of freedom. Gauge symmstry exists only in the limit $\kappa \to \infty$ when the constraint becomes $\sum s_I M_I = \sum \tilde{M}_I$.

The ultrarelativistic limit corresponds to $w_I^2 = A_I + \kappa^2$, $\tilde{w}_I^2 = \tilde{A}_I + \kappa^2$ with $\kappa^2 \to \infty$. The parameter ϵ is of the order $\frac{1}{\kappa}$, and $t \simeq \tau/\kappa$. There is a subtlety with these exact worldsheet solutions; we do not know if the solutions with different κ belong to the same λ-family.

3.5 Integrability of the Neumann system (a very brief review)

Here we will review the basic properties of the Neumann system, following mostly [20]. Consider the dynamical system with the lagrangian

$$L = \frac{1}{2} \sum_{\mu=1}^{N} s_\mu \left((\dot{x}_\mu)^2 - a_\mu x_\mu^2 \right) \tag{3.18}$$

where $s_\mu = \pm 1$ and $\sum s_\mu x_\mu^2 = 1$. There are $N-1$ independent integrals of motion:

$$F_\mu = x_\mu^2 + \sum_{\nu \neq \mu} s_\nu \frac{(x_\mu \dot{x}_\nu - x_\nu \dot{x}_\mu)^2}{a_\mu - a_\nu} \tag{3.19}$$

$$\sum s_\mu F_\mu = 1\,. \tag{3.20}$$

Introduce the parametrization:

$$x_\mu^2 = s_\mu \frac{\prod_\nu (t_\nu - a_\mu)}{\prod_{\lambda\neq\mu}(a_\lambda - a_\mu)}\,. \tag{3.21}$$

Using the identities:

$$\sum_\nu \frac{s_\nu x_\nu^2}{t_\rho - a_\nu} = 0 \tag{3.22}$$

$$\sum_\nu \frac{s_\nu x_\nu^2}{(t_\rho - a_\nu)^2} = \frac{\prod_{\sigma\neq\rho}(t_\rho - t_\sigma)}{\prod_\nu (t_\rho - a_\nu)} \tag{3.23}$$

we can rewrite the conserved quantities in t-coordinates:

$$F_\mu(t,\dot t) = x_\mu^2 \left(1 - \frac{1}{4}\sum_\rho \frac{\dot t_\rho^2}{t_\rho - a_\mu}\frac{\prod_{\sigma\neq\rho}(t_\rho - t_\sigma)}{\prod_\nu (t_\rho - a_\nu)}\right). \tag{3.24}$$

The trajectories are determined by F_μ =const. They depend on $N-1$ constants $\{b_1,\ldots,b_{N-1}\}$:

$$\frac{dt_\mu}{d\sigma} = 2\varepsilon_\mu \frac{\sqrt{\prod_\kappa (t_\mu - a_\kappa)\prod_\lambda (t_\mu - b_\lambda)}}{\prod_{\nu\neq\mu}(t_\mu - t_\nu)} \tag{3.25}$$

Here $\varepsilon_\mu = \pm 1$. On these trajectories $F_\mu = \frac{\prod_\lambda (a_\mu - b_\lambda)}{\prod_{\nu\neq\mu}(a_\mu - a_\nu)}$.

We are interested in the special case of the Neumann problem, when N is even and pairs of a_μ coincide. Consider the limit $a_{2I}\to a_{2I-1} = A_I$ with $t_{\frac{N}{2}+I-1}$ locked between a_{2I} and a_{2I-1} for $I=1,\ldots,N/2$. For the remaining t_I with $I = 1,\ldots,N/2-1$ we get:

$$\frac{dt_I}{d\sigma} = 2\varepsilon_I \frac{\sqrt{\prod_k (t_I - b_k)}}{\prod_{K\neq I}(t_I - t_K)}\,. \tag{3.26}$$

The trajectories of t_I do not depend on A_I. To describe the oscillations of the coordinates trapped between a_{2I} and a_{2I-1} we introduce the angles θ_I:

$$\tan\theta_I = \sqrt{\frac{a_{2I} - t_{N/2+I-1}}{t_{N/2+I-1} - a_{2I-1}}}\,. \tag{3.27}$$

Denote $X_I = x_{2I-1} + i x_{2I}$. We have: $X_I = |X_I| e^{i\theta_I}$ and $|X_I|^2 = \frac{\prod_K (A_I - t_K)}{\prod_{J\neq I}(A_I - A_J)}$. The angles are cyclic variables:

$$\frac{d\theta_I}{d\sigma} = \varepsilon_{N/2+I-1}\frac{\sqrt{-\prod_j (A_I - b_j)}}{\prod_J (A_I - t_J)}\,. \tag{3.28}$$

The corresponding momentum is conserved:

$$M_I = |x_I|^2 \frac{d\theta_I}{d\sigma} = \varepsilon_{N/2+I-1}\frac{\sqrt{-\prod_j (A_I - b_j)}}{\prod_{J\neq I}(A_I - A_J)}\,. \tag{3.29}$$

We can use the identity $\frac{1}{\prod_J (A-t_J)} = \sum_J \left[\frac{1}{\prod_{K\neq J}(t_J-t_K)}\right]\frac{1}{A-t_J}$ and write:

$$\theta_I = \varepsilon_{N/2+I-1}\sqrt{-\prod_j (A_I - b_j)}\sum_J \int \frac{\varepsilon_J dt_J}{(A_I - t_J)\sqrt{\prod_k (t_J - b_k)}}\,. \tag{3.30}$$

It is convenient to put $\varepsilon_{N/2+I-1} = 1$ by replacing X_I with its complex conjugate if $\varepsilon_{N/2+I-1} = -1$.

3.6 The product of two Neumann systems

The motion on $(AdS_5 \times S^5)/V$ is described by the product of two Neumann systems. The projection to AdS_5 is described by the Neumann system with the parameters (A_0, A_1, A_2) and the action variables $b_1, \ldots, b_5$. The complex coordinates (Y_0, Y_1, Y_2) are parametrized by $t_1, t_2, \theta_0, \theta_1, \theta_2$:

$$Y_I = |Y_I|e^{i\theta_I}\,, \qquad |Y_I|^2 = s_I \frac{\prod_J (t_J - A_I)}{\prod_{K\neq I}(A_K - A_I)} \tag{3.31}$$

where s_I is a sign: $(s_0, s_1, s_2) = (1, -1, -1)$. The projection of the trajectory on AdS_5 is described by eqs. (3.26) and (3.28). The projection to S^5 is described by the Neumann system with the parameters $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$; the action variables are $\tilde{b}_1, \ldots, \tilde{b}_5$. The coordinates on the sphere are:

$$Z_I = |Z_I|e^{i\tilde{\theta}_I}\,, \qquad |Z_I|^2 = \frac{\prod_J (\tilde{t}_J - \tilde{A}_I)}{\prod_{K\neq I}(\tilde{A}_K - \tilde{A}_I)}\,. \tag{3.32}$$

The projection of the trajectory on S^5 is described by eqs. (3.26) and (3.28) with $(t, \theta, A, b) \to (\tilde{t}, \tilde{\theta}, \tilde{A}, \tilde{b})$.

We are interested in the periodic trajectories, with the period $\int d\sigma = 2\pi$. The Neumann system is integrable, therefore the phase space is foliated by invariant tori [22]. The trajectory in the phase space is $\dot{\phi}_a = \omega_a(I)$ where ϕ_a are the angle variables and I are the action variables specifying the torus. For the trajectory to be periodic with the period 2π the frequencies ω_a should be integers:

$$\omega_a(I) = m_a \tag{3.33}$$

These are the equations on the action variables. The number of unknowns is equal to the number of equations, therefore we expect in general to have a discrete set of periodic trajectories, corresponding to the special values of the action variables. We will say that the torus is periodic if the frequencies are integer. Let us see how it works in our case.

To make the formulas more transparent we consider the system corresponding to $AdS_{N-1} \times S^{N-1}$ keeping in mind that we are mostly interested in $N = 6$. We are interested in the solutions which satisfy the constraint $\sum s_I M_I = \sum \tilde{M}_I$ and are periodic modulo the overall phase. The constraint reads:

$$\sum_{I=0}^{N-1} s_I \frac{\sqrt{-\prod_j (A_I - b_j)}}{\prod_{J\neq I}(A_I - A_J)} = \sum_{I=1}^{N} \frac{\sqrt{-\prod_j (\tilde{A}_I - \tilde{b}_j)}}{\prod_{J\neq I}(\tilde{A}_I - \tilde{A}_J)}\,. \tag{3.34}$$

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To formulate the periodicity conditions for t_I and $\tilde{t}_I$ we rewrite the equations (3.26) in the following form:

$$\sum_I \frac{\varepsilon_I t_I^k dt_I}{\sqrt{\prod_j (t_I - b_j)}} = 0\,, \qquad \text{for } k = 0, \dots, \frac{N}{2} - 3 \tag{3.35}$$

$$\sum_I \frac{\varepsilon_I t_I^{N/2-2} dt_I}{\sqrt{\prod_j (t_I - b_j)}} = 2\, d\sigma\,. \tag{3.36}$$

We can consider $N/2-1$ pairs (t_I, ε_I) as specifying $N/2-1$ points on the curve $\mathcal{C}$ described by the equation $y^2 = \prod_{j=1}^{N-1}(t - b_j)$. Indeed, t_I gives the value of t and ε_I fixes the sign of y. The periodic trajectory then defines a homology class $c \in H_1(\mathcal{C}, \mathbf{Z})$ as the formal sum of the trajectories of the points (t_I, ε_I) for $I = 1, \dots, N/2-1$. We have

$$\oint_c \frac{\varepsilon t^k dt}{\sqrt{\prod_j (t - b_j)}} = \oint_{\tilde{c}} \frac{\tilde{\varepsilon} \tilde{t}^k d\tilde{t}}{\sqrt{\prod_j (\tilde{t} - \tilde{b}_j)}} = 0 \qquad \text{for } k = 0, \dots, \frac{N}{2} - 3\,, \tag{3.37}$$

$$\oint_c \frac{\varepsilon t^{N/2-2} dt}{\sqrt{\prod_j (t - b_j)}} = \oint_{\tilde{c}} \frac{\tilde{\varepsilon} \tilde{t}^{N/2-2} d\tilde{t}}{\sqrt{\prod_j (\tilde{t} - \tilde{b}_j)}} = 4\pi\,. \tag{3.38}$$

It turns out that the converse is also true [21]. The cycle $(c, \tilde{c}) \in H_1(\mathcal{C} \times \tilde{\mathcal{C}}, \mathbf{Z})$ satisfying (3.37), (3.38) defines a periodic (modulo cyclic variables) trajectory of the product of two Neumann systems with the period 2π. Therefore (3.37) and (3.38) are the periodicity conditions for the t variables. It is interesting that they do not depend on A_I. We have to also make sure that the cyclic variables $\theta, \tilde{\theta}$ are periodic functions of σ. The periodicity conditions for $\theta, \tilde{\theta}$ read:

$$\sqrt{-\prod_j (A_I - b_j)} \oint_c \frac{\varepsilon dt}{(A_I - t)\sqrt{\prod_k (t - b_k)}} = 2\pi m_I + \mu \tag{3.39}$$

$$\sqrt{-\prod_j (\tilde{A}_I - \tilde{b}_j)} \oint_{\tilde{c}} \frac{\varepsilon d\tilde{t}}{(\tilde{A}_I - \tilde{t})\sqrt{\prod_k (\tilde{t} - \tilde{b}_k)}} = 2\pi \tilde{m}_I + \mu \tag{3.40}$$

where $\mu \in \mathbf{R}$ is an undetermined overall phase.

It is convenient to think about not just one Neumann system, but the whole family of integrable systems with different values of A_I and $\tilde{A}_I$. We have $N/2$ of A_I and $N/2$ of $\tilde{A}_I$, but both A and $\tilde{A}$ are defined up to a common shift. Therefore we have $N-2$ parameters in the integrable lagrangian. For the given lagrangian, we have $N-1$ of b and $N-1$ of $\tilde{b}$, the total of $2N-2$ parameters specifying the invariant torus. The total number of parameters is therefore $3N-4$. Let us count the constraints. We have one constraint (3.34) that the trajectory is orthogonal to V, $N/2-1$ for the periodicity of t_I, $N/2-1$ for the periodicity of $\tilde{t}_I$, and $N-1$ for the periodicity of θ_I and $\tilde{\theta}_I$ modulo an overall phase μ. The total number of constraints is therefore $2N-2$.

Thus we expect that in the space of parameters $A, \tilde{A}$ and action variables $b, \tilde{b}$ there is the $N-2$ dimensional subspace corresponding to the periodic trajectories. The dimension of this subspace coincides with the number of independent charges $q_A = \int_0^{2\pi} d\sigma |X_A|^2$.

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Based on this counting of parameters, we expect to be able to adjust the parameters to get a periodic trajectory with prescribed values of q_A. The kinetic part of the action will then give the one-loop anomalous dimension.

3.7 Special solutions

The periodic tori may degenerate. For example suppose that $b_1 \to b_2$ and t_1 is trapped between b_1 and b_2. Then (3.26) and (3.28) imply that b_1, b_2 and t_1 decouple from the equations for $t_2, \ldots, t_{N/2-1}$ and $\theta_2, \ldots, \theta_{N/2-1}$. We do not have to impose the periodicity condition on t_1, because in the limit $b_1 \to b_2$ we have t_1 =const. In fact the value of $b_1 = b_2$ does not enter in the remaining periodicity conditions (it does however enter the constraint (3.34).) Therefore we have one less parameter (because we impose $b_1 = b_2$) but also one less constraint. This means that we still have the $N-2$-parameter family of periodic trajectories corresponding to $b_1 = b_2$. We conclude that the space of periodic trajectories consists of several branches, corresponding to degenerations of the periodic tori.

An interesting special case corresponds to $N-2$ of the parameters b coinciding pairwise. Suppose that $b_{2I} \to b_{2I-1} = B_I$ for $I = 1, \ldots, N/2-1$ and use the shift symmetry to put $b_{N-1} = 0$. Consider t_J oscillating between b_{2J-1} and b_{2J}. From (3.28) we get the periodicity conditions:

$$A_I = (m_I + \mu)^2\,, \qquad \tilde{A}_I = (\tilde{m}_I + \mu)^2 \tag{3.41}$$

where m_I are integers and μ is real. The oscillation of t_J between b_{2J-1} and b_{2J} is described by the equation:

$$\frac{d\phi_I}{d\sigma} = \sqrt{B_I}\,. \tag{3.42}$$

Let us put $B_I = n_I^2$. The n_I does not have to be an integer, because the coordinate ϕ_I degenerates when $b_{2J-1} \to b_{2J}$. But if n_I is integer then we can resolve B_I into a pair $b_{2I-1} \neq b_{2I}$. This means that the periodic trajectories with integer n_I are at the intersection of the different branches. Independently of whether or not n_I are integer we can explicitly write down the corresponding periodic trajectories. The absolute values of Z_I and Y_I are constant, and the phases depend on σ linearly:

$$Y_I(\sigma) = e^{i(m_I+\mu)\sigma}|Y_I|\,, \qquad Z_I(\sigma) = e^{i(\tilde{m}_I+\mu)\sigma}|Z_I| \tag{3.43}$$

$$|Y_I|^2 = \frac{1}{2\pi}q_I = s_I \frac{\prod_J[(m_I+\mu)^2 - n_J^2]}{\prod_{J\neq I}[(m_I+\mu)^2-(m_J+\mu)^2]} \tag{3.44}$$

$$|Z_I|^2 = \frac{1}{2\pi}\tilde{q}_I = \frac{\prod_J[(\tilde{m}_I+\mu)^2 - \tilde{n}_J^2]}{\prod_{J\neq I}[(\tilde{m}_I+\mu)^2-(\tilde{m}_J+\mu)^2]}\,. \tag{3.45}$$

The momenta are:

$$M_I = s_I(m_I+\mu)\frac{\prod_J[(m_I+\mu)^2 - n_J^2]}{\prod_{J\neq I}[(m_I+\mu)^2-(m_J+\mu)^2]} \tag{3.46}$$

$$\tilde{M}_I = (\tilde{m}_I+\mu)\frac{\prod_J[(\tilde{m}_I+\mu)^2 - \tilde{n}_J^2]}{\prod_{J\neq I}[(\tilde{m}_I+\mu)^2-(\tilde{m}_J+\mu)^2]}\,. \tag{3.47}$$

The action is:

$$S = \sum (m_I + \mu)^2 - \sum n_J^2 + \sum (\tilde{m}_I + \mu)^2 - \sum \tilde{n}_J^2 \tag{3.48}$$

and the constraint $\sum s_I M_I = \sum \tilde{M}_I$ reads:

$$\begin{aligned} &-(m_0 + \mu)|Y_0|^2 + (m_1 + \mu)|Y_1|^2 + (m_2 + \mu)|Y_2|^2 + \\ &+(\tilde{m}_1 + \mu)|Z_1|^2 + (\tilde{m}_2 + \mu)|Z_2|^2 + (\tilde{m}_3 + \mu)|Z_3|^2 = 0 \,. \end{aligned} \tag{3.49}$$

One can see that μ drops out of this constraint (enters only through $|Z_I|^2$ and $|Y_I|^2$) because $\sum s_I |Y_I|^2 = \sum |Z_I|^2$. Therefore the constraint actually imposes a restriction on the possible values of the R-charges and the spins:

$$\sum s_I m_I q_I = \sum \tilde{m}_I \tilde{q}_I \,. \tag{3.50}$$

This restriction on charges and spins is a feature of the "totally degenerate" periodic trajectories. For generic $n_I, \tilde{n}_I$ the small deformations of these totally degenerate trajectories correspond to varying $n_I, \tilde{n}_I$ in (3.44), (3.45). But if some of the $n_I, \tilde{n}_I$ are integer, then there are additional deformations corresponding to splitting the corresponding pair of coinciding b_j. It would be interesting to understand what is special about these "integer" values of the R-charge and the spin in the dual gauge theory.

Another special case is when $(b_1, b_2, b_3) \to (A_0, A_1, A_2)$ and $(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) \to (\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$. In this case all the angles θ_I, $\tilde{\theta}_I$ are frozen. It is the same as restricting Y_I and Z_I to be real. This case was considered in [11].

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